Supersymmetry, factorisation of the Schrodinger equation and a Hamiltonian hierarchy

1985 J. Phys. A: Math. Gen. 18 L57

(http://iopscience.iop.org/0305-4470/18/2/001)

J. Phys. A: Math. Gen. **18** (1985) L57–L61. Printed in Great Britain

LETTER TO THE EDITOR

Supersymmetry, factorisation of the Schrödinger equation and a Hamiltonian hierarchy

C V Sukumar

Hahn-Meitner Institut fur Kernforschung, Postfach 39-01-28, D1000 Berlin 39, West Germany

Received 25 October 1984

Abstract. We present a systematic procedure for constructing a hierarchy of non-relativistic Hamiltonians with the property that the adjacent members of the hierarchy are 'supersymmetric partners' i.e. they share the same eigenvalue spectrum except for the 'missing' ground state and the eigenvectors are simply related.

Consider a Hermitian positive semi-definite operator of the form $H = A^+A^-$ in which A^+ is the Hermitian adjoint of the operator A^-. Let Ψ be an eigenfunction of H with eigenvalue E. The eigenvalue equation

$$A^+A^-\Psi = E\Psi \tag{1}$$

leads, on multiplication from the left by A^-, to

$$A^-A^+(A^-\Psi) = E(A^-\Psi). \tag{2}$$

Equations (1) and (2) lead to the following theorem.

Theorem 1. An eigenvalue of the operator A^+A^- is also an eigenvalue of the operator A^-A^+, except when $A^-\Psi = 0$. The normalised eigenfunctions of A^+A^- and A^-A^+, denoted by Ψ and φ respectively, are connected by the equations

$$\varphi = E^{-1/2}A^-\Psi, \qquad \Psi = E^{-1/2}A^+\varphi. \tag{3}$$

Bernstein and Brown (1984) considered a Hamiltonian of the form $H_+ = A^+A^-$ with $A^\pm = (\mp\partial/\partial x + \frac{1}{2}\partial U/\partial x)$ for a specified function $U(x)$. They showed that the scalar Hamiltonian H_+ and its 'partner' $H_- = A^-A^+$, corresponding to the potentials $V_\pm = (\frac{1}{2}\partial U/\partial x)^2 \pm \frac{1}{2}\partial^2 U/\partial x^2$, can be viewed as the 'bosonic' and the 'fermionic' components of a supersymmetric Hamiltonian

$$\mathscr{H} = [(-\partial^2/\partial x^2 + W^2)I + \sigma_3\partial W/\partial x] \tag{4}$$

in which $W = -\frac{1}{2}\partial U/\partial x$, I is the unit matrix and σ_3 is the Pauli spin matrix (Witten 1981). Since H_+ has a ground state with eigenvalue $E = 0$ and an eigenfunction that satisfies $A^-\Psi = 0$, theorem 1 implies the following mapping of the eigenvalues of H_- and H_+:

$$E_-^{(n)} = E_+^{(n+1)} \qquad n = 0, 1, 2, \ldots. \tag{5}$$

0305-4470/85/020057+05\$02.25

Bernstein and Brown (1984) were thus able to infer the energy of the first excited state of H_+ by calculating the ground state energy of H_-.

In this letter we consider the non-relativistic Hamiltonian $H = -\frac{1}{2}\partial^2/\partial x^2 + V(x)$ for any potential $V(x)$ that can support at least one bound state. We factorise H in the form

$$H = -\tfrac{1}{2}\partial^2/\partial x^2 + V(x) \equiv A^+A^- + \varepsilon \tag{6}$$

where

$$A^\pm = (1/\sqrt{2})(\pm\partial/\partial x + \tilde{V}(x)). \tag{7}$$

The unknown function $\tilde{V}(x)$ and the undetermined constant ε are then determined by the consistency condition that

$$\tilde{V}^2 + \partial\tilde{V}/\partial x = 2(V - \varepsilon). \tag{8}$$

This condition is clearly satisfied if

$$\tilde{V} = (1/\Psi^{(0)})(\partial\Psi^{(0)}/\partial x) \qquad \text{and} \qquad \varepsilon = E^{(0)} \tag{9}$$

where $\Psi^{(0)}$ and $E^{(0)}$ are the ground state eigenfunction and eigenvalue of H. The choice of the wavefunction in equation (9) is motivated by the consideration that A^+A^- is required to be a positive semi-definite operator with eigenvalues $\geqslant 0$. This leads to the following theorem.

Theorem 2. Any Hamiltonian of the form $H = -\frac{1}{2}\partial^2/\partial x^2 + V(x)$, which has a ground state $(\Psi^{(0)}, E^{(0)})$ can be factorised as $H = A^+A^- + E^{(0)}$ with $A^\pm = (1/\sqrt{2})[\pm\partial/\partial x + (1/\Psi^{(0)})(\partial\Psi^{(0)}/\partial x)]$.

We now show that theorems 1 and 2 enable the generation of a hierarchy of Hamiltonians with simple relations connecting the eigenvalues and eigenfunctions of the different members of the hierarchy. Starting from a Hamiltonian H_1 for a potential $V_1(x)$ that can support M bound states with a ground state $(\Psi_1^{(0)}, E_1^{(0)})$ and applying theorem 2 we get

$$H_1 = -\tfrac{1}{2}\partial^2/\partial x^2 + V_1(x) \equiv A_1^+A_1^- + E_1^{(0)} \tag{10}$$

$$A_1^\pm = (1/\sqrt{2})[\pm\partial/\partial x + (1/\Psi_1^{(0)})(\partial\Psi_1^{(0)}/\partial x)]. \tag{11}$$

We can now construct a 'supersymmetric partner' H_2 with potential $V_2(x)$ given by

$$H_2 = A_1^-A_1^+ + E_1^{(0)} \equiv -\tfrac{1}{2}\partial^2/\partial x^2 + V_2(x) \tag{12}$$

$$V_2(x) = V_1(x) + [A_1^-, A_1^+] = V_1(x) - (\partial^2/\partial x^2)\ln\Psi_1^{(0)}. \tag{13}$$

Since $A_1^-\Psi_1^{(0)} = 0$, theorem 1 then shows that the spectra of H_1 and H_2 satisfy the condition that

$$E_2^{(n)} = E_1^{(n+1)} \qquad n = 0, 1, 2, \ldots, M-2 \tag{14}$$

and the normalised eigenfunctions of H_1 and H_2 are connected by the equation

$$\Psi_2^{(n)} = [E_1^{(n+1)} - E_1^{(0)}]^{-1/2}A_1^-\Psi_1^{(n+1)}. \tag{15}$$

By applying theorem 2 to the new Hamiltonian H_2 we can refactorise H_2 in terms of its ground state $(\Psi_2^{(0)}, E_2^{(0)})$ as

$$H_2 = -\tfrac{1}{2}\mathrm{d}^2/\mathrm{d}x^2 + V_2(x) \equiv A_2^+A_2^- + E_2^{(0)} \tag{16}$$

$$A_2^\pm = (1/\sqrt{2})[\pm\partial/\partial x + (1/\Psi_2^{(0)})(\partial\Psi_2^{(0)}/\partial x)]. \tag{17}$$

This new factorisation of H_2 in turn leads to a new 'supersymmetric partner' H_3 given by

$$H_3 = A_2^- A_2^+ + E_2^{(0)} \tag{18}$$

whose spectrum can be determined by the application of theorem 1.

By repeated application of theorems 1 and 2 we can thus generate a hierarchy of Hamiltonians given by

$$H_n = -\tfrac{1}{2}\partial^2/\partial x^2 + V_n(x) \equiv A_n^+ A_n^- + E_n^{(0)} = A_{n-1}^- A_{n-1}^+ + E_{n-1}^{(0)} \tag{19}$$

$$A_n^{\pm} = (1/\sqrt{2})(\pm\partial/\partial x + (1/\Psi_n^{(0)})(\partial\Psi_n^{(0)}/\partial x)] \tag{20}$$

$$V_n(x) = V_{n-1}(x) - (\partial^2/\partial x^2)\ln\Psi_{n-1}^{(0)} = V_1(x) - (\partial^2/\partial x^2)\ln(\Psi_1^{(0)}\Psi_2^{(0)}\ldots\Psi_{n-1}^{(0)}),$$
$$n = 2, 3, \ldots, M \tag{21}$$

whose spectra satisfy the conditions

$$E_n^{(m)} = E_{n-1}^{m+1} = \cdots = E_1^{n+m-1}, \qquad \begin{array}{l} m = 0, 1, 2, \ldots, M-n \\ n = 2, 3, \ldots, M \end{array} \tag{22}$$

$$\Psi_n^{(m)} = \{[E_1^{(n+m-1)} - E_1^{(n-1)}][E_1^{(n+m-1)} - E_1^{(n-2)}]\ldots[E_1^{(n+m-1)} - E_1^{(0)}]^{-1/2}$$
$$\times A_{n-1}^- A_{n-2}^- \ldots A_1^- \Psi_1^{(n+m-1)} \tag{23}$$

in which A_{n-1}^- is the annihilation operator of the ground state in the potential $V_{n-1}(x)$.

We have outlined a simple procedure for constructing a hierarchy of Hamiltonians with the property that the Nth member of the hierarchy has the same eigenvalue spectrum as the first member H_1 except for missing the first $(N-1)$ eigenvalues of H_1. In particular, the Nth excited state of H_1 is degenerate with the ground state of H_{N+1} and the corresponding wavefunctions are simply related. We give some examples of this hierarchy.

Harmonic oscillator

The potential $V_1 = \frac{1}{2}\omega^2 x^2$ with the ground state wavefunction $\Psi_1^{(0)} \sim e^{-\omega x^2/2}$ leads to the hierarchy

$$V_n(x) = V_1(x) + (n-1)\omega. \tag{24}$$

The Hamiltonian hierarchy corresponds to shifting the entire potential well up in energy in units of $\hbar\omega$ ($\hbar = 1$ in the units we have adopted).

Particle in a box

The potential

$$\begin{array}{ll} V_1 = 0 & |x| < a \\ V_1 = \infty & |x| = a \end{array} \tag{25}$$

with the eigenvalue spectrum

$$E_1^{(m)} = (\pi^2/8a^2)(m+1)^2, \qquad m = 0, 1, 2, \ldots \tag{26}$$

has the ground state $\Psi_1^{(0)} \sim \cos(\pi x/2a)$ and leads to

$$V_n(x) = V_1(x) + (\pi^2/8a^2)n(n-1)\sec^2(\pi x/2a) \tag{27}$$

$$E_n^{(m)} = E_1^{(n+m-1)} = (\pi^2/8a^2)(n+m)^2, \qquad \begin{array}{l} m = 0, 1, 2, \ldots \\ n = 1, 2, 3, \ldots . \end{array} \tag{28}$$

Thus the potential for a particle in a box generates a series of $\sec^2(\pi x/2a)$ potentials with increasing strengths. Indeed the spectrum of $\sec^2 x$ potential is calculated in Morse and Feshbach (1953) to be of the form J^2, the starting value of the integer J being determined by the strength of the potential.

We now consider examples of Hamiltonians with both discrete and continuous states.

Hydrogen atom

The s-wave radial equation corresponds to the potential $V_1(r) = -b/r$ with the spectrum

$$E_1^{(m)} = -b^2/2(m+1)^2 \qquad m = 0, 1, 2, \ldots . \tag{29}$$

The ground state wavefunction $\Psi_1^{(0)} \sim r\mathrm{e}^{-br}$ leads to

$$V_2(r) = V_1 + 1/r^2. \tag{30}$$

V_2 thus corresponds to the potential for the p-wave radial equation with the ground state $\Psi_2^{(0)} \sim r^2\mathrm{e}^{-br/2}$ and the spectrum

$$E_2^{(m)} = -b^2/2(m+2)^2, \qquad m = 0, 1, 2, \ldots . \tag{31}$$

The potential $V_n(r)$ can be shown to be given by

$$V_n(r) = -(b/r) + n(n-1)/2r^2, \qquad n = 1, 2, \ldots \tag{32}$$

with the spectrum

$$E_n^{(m)} = -b^2/2(n+m)^2, \qquad m = 0, 1, 2, \ldots . \tag{33}$$

The Hamiltonian hierarchy now corresponds to the addition of the centrifugal potential. Through equation (22) we recover the well known property that the *Ns* state of an attractive Coulomb potential is degenerate with the *Np*, *Nd*... states.

Morse potential

The potential

$$V_1(x) = D_1 \exp[2(x_1 - x)/d] - 2D_1 \exp[(x_1 - x)/d] \tag{34}$$

which supports a finite number M of bound states has the spectrum (Morse and Feshbach 1953)

$$E_1^{(m)} = -\alpha_1^2/2d^2 + (m+\tfrac{1}{2})\alpha_1/d^2 - 2d^{-2}(m+\tfrac{1}{2})^2, \qquad m = 0, 1, \ldots, M-1 \tag{35}$$

in which $\alpha_1 = d\sqrt{2D}$ and M is the largest integer less than $(\alpha_1 + \frac{1}{2})$. The ground state wavefunction

$$\Psi_1^{(0)}(x) \sim \exp[-(\alpha_1 - \tfrac{1}{2})x/d - \alpha_1 \exp(x_1 - x)/d] \tag{36}$$

leads to

$$V_2(x) = V_1(x) + \alpha_1 \exp[(x_1 - x)/d]. \tag{37}$$

$V_2(x)$ corresponds to a new Morse potential with the parameters $\alpha_2 = (\alpha_1 - 1)$, $x_2 = x_1 + d \ln(\alpha_1/\alpha_1 - 1)$ and the spectrum

$$E_2^{(m)} = E_1^{(m+1)} \qquad m = 0, 1, 2, \ldots, M-2. \tag{38}$$

An invariance property of the spectrum of the Morse potential is now apparent since the substitution $\alpha_2 = \alpha_1 - 1$ indeed leads to the equality $E_1^{(m)}(\alpha_2) = E_1^{(m+1)}(\alpha_1)$. The $(M+1)$ member Hamiltonian hierarchy thus corresponds to a set of Morse potentials with differing α's, $\alpha_N = \alpha_1 - (N-1)$ and differing minima x_N. The Mth member of the hierarchy supports a single bound state and the condition $M < (\alpha_1 + \frac{1}{2})$ then leads to a V_{n+1} that is a Morse potential which is not deep enough to support a bound state.

We have illustrated the Hamiltonian hierarchy in equation (19) with some analytically soluble examples. Clearly the procedure outlined in this letter is very general. We will show in a subsequent detailed publication a generalisation of this approach to Hamiltonian functions of an arbitrary number of variables. The Gelfand-Levitan equations in inverse scattering theory also produce families of potentials which differ by the presence of additional bound states (Abraham and Moses 1980, Nieto 1984). The connection of the approach adopted in this letter to the Gelfand-Levitan equations will be discussed in a future publication.

I thank D M Brink, B Buck, M A Nagarajan and R Baldock for useful discussions.

Note added in proof. Since this letter was submitted for publication we have become aware of the publication of A A Andrianov, N V Borisov and M V Ioffe in *Phys. Lett.* **105A** 19-22.

References

Abraham P B and Moses H E 1980 *Phys. Rev.* A **22** 1333-40
Bernstein M and Brown L S 1984 *Phys. Rev. Lett.* **52** 1933-5
Morse P and Feshbach H 1953 *Methods of Theoretical Physics* (New York: McGraw Hill) vol 1, p768, vol 2, p 1672
Nieto M M 1984 *Phys. Lett.* **145B** 208-10
Witten E 1981 *Nucl. Phys.* B **188** 513-54